Evolution of a scalar gradient's probability density function in a random flow

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The evolution of the probability density function (PDF) of the passive scalar gradient is studied in the limit of large Peclet and Prandtl numbers in d dimensions. Without diffusion, the closed Fokker-Planck equation can be derived and solved analytically leading to a number of conclusions. In particular, it allows the description of the restoration of the rotational symmetry and enables one to distinguish different regimes of evolution and different intervals of PDF behavior. [S1063-651X(98)12509-6]

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Recently the problem of advection of a passive scalar in a random smooth velocity field (Batchelor regime) has attracted a lot of attention due to the fact that a number of questions can be treated analytically. Such a situation is very rare in the theory of turbulence and this is why the model gains its importance. Some of the answers obtained seem to be of a general nature. In this paper we address the question of the gradients' probability density function (PDF) evolution. Generally, this question is very complicated because one cannot write a closed equation for the PDF. However, if one starts with an almost uniform distribution of the passive scalar, clearly there exists the time period when diffusion is ineffective for the main portion of the PDF. During this stage of evolution one may discard the diffusion term and the corresponding Fokker-Planck equation can be derived, furnishing us with a wealth of information about the PDF's evolution as we shall show below. Besides, the information about the influence of velocity and pumping terms on statistics is obtained. For the Batchelor regime, comparison with the stationary case is possible [1], showing that diffusion dramatically modifies the PDF everywhere, which, in particular, implies that the "adiabatic approach" [2] is wrong for $d>1$. The reason for this is that in $d=1$ the regions of small and large gradients are separated so that the diffusion can be neglected in the consideration of the PDF at small gradients. Conversely, at $d>1$ due to incompressibility scalar pieces that have small gradients in one direction will possess large gradients in a transverse direction, making it necessary to include diffusion for an adequate description. We show that the Fokker-Planck equation in Fourier space can be put after a substitution in a form that allows us to solve the Cauchy problem in quadratures. We derive the asymptotic forms of the solution and show that they have a clear physical meaning, thus reconstructing the qualitative picture of the evolution.

The equation governing the advection of passive scalar θ by the velocity field $\boldsymbol{v}$ in the presence of external source f is

$$(\partial_t+\boldsymbol{v}\cdot\boldsymbol{\nabla}-\kappa\nabla^2)\theta=f,\quad \boldsymbol{\nabla}\cdot\boldsymbol{v}=0. \tag{1}$$

We consider this within the framework of the Kraichnan model, in which both velocity and pumping statistics are Gaussian, isotropic, and δ-correlated in time, the correlation function for the pumping being $\langle f(t_1,\boldsymbol{r}_1)f(t_2,\boldsymbol{r}_2)\rangle=\chi(r_{12})\delta(t_1-t_2)$, where $\chi(r_{12})$ is a function of $r_{12}=|\boldsymbol{r}_1-\boldsymbol{r}_2|$ and decays on the scale L. The velocity field is considered to be smooth in space. It can be characterized by the correlation function of the strain $\sigma_{ij}=\partial_j v_i$, which in this case equals

$$\langle\sigma_{ij}(t)\sigma_{kl}(t')\rangle=\Theta[(d+1)\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}-\delta_{ij}\delta_{kl}]\delta(t-t'),$$

where Θ is a constant of proportionality and d the dimension of the space. The possibility of approximating velocity by the first two terms in the Taylor expansion means that the pumping scale L is smaller than L_u, the velocity dissipation scale. To have our scalar field fully turbulent, we demand its mean diffusion scale r_d to satisfy $L_u\gg L\gg r_d$. Because $r_d\sim Pr^{-1/2}L_u$ with the Prandtl number $Pr=\nu/\kappa$, this implies $Pr\gg1$. Finally, since the Peclet number $Pe^2=d\Theta L^2/\kappa$ equals the ratio between pumping and diffusion scales, we have $Pe\gg1$. It is worth mentioning that this situation is physically realizable. Such materials as glycerin and oils possess Prandtl numbers of the order 10^4.

Let us now consider Eq. (1) without the diffusion term. The equation for $\boldsymbol{w}\equiv\nabla\theta$ follows:

$$\partial_t w_i=-(\boldsymbol{v}\cdot\boldsymbol{\nabla})w_i-\sigma_{ji}w_j+\partial_i f. \tag{2}$$

Using it, the closed Fokker-Planck equation for the scalar gradient's PDF $P(\boldsymbol{w},t)\equiv\langle\delta(\boldsymbol{w}(\boldsymbol{r},t)-\boldsymbol{w})\rangle$ can be derived:

$$\partial_t P=\frac{B}{2}\nabla_w^2 P+\frac{\Theta}{2}\left((d+1)w^2\nabla_w^2-2w_i w_j\frac{\partial^2}{\partial w_i\partial w_j}\right)P, \tag{3}$$

where $B=-\chi''(0)$. It is of interest to understand the influence of the velocity and force terms separately before combining them together. The first term, the force one, corresponds to the random walk for $\boldsymbol{w}$, whereas, as we show below, the velocity term describes the statistics of the distance between two points advected by a random velocity field. Indeed, let us consider the statistics of the distance $\boldsymbol{r}$ between two points in the fluid, the dynamical equation being $\dot{\boldsymbol{r}}=\sigma\boldsymbol{r}$. Due to the fact that the strain statistics is invariant under $\sigma\to-\sigma^t$, one concludes that $G(\boldsymbol{r},t)\equiv\langle\delta(\boldsymbol{r}(t)-\boldsymbol{r})\rangle$ satisfies the same equation (3) with the force term omitted. To solve this equation we first consider the radial part:

$$\partial_t G=D[r^2\partial_r^2+(d+1)r\partial_r]G, \tag{4}$$

with $D\equiv\Theta(d-1)/2$ and G satisfying the normalized initial condition $G(r,0)=\Gamma(d/2)/(2\pi^{d/2}r'^{\,d-1})\delta(r-r')$. Changing variables $r=e^z$, the equation is easily solved giving [3],

$$G(r,t)=\frac{1}{\sqrt{\pi Dt}}\frac{\Gamma(d/2)}{4\pi^{d/2}r'^{d}}\exp\left[-\frac{(\ln[r/r']+Ddt)^2}{4Dt}\right]. \tag{5}$$

Now we consider the full problem of two fluid particles initially separated by $\boldsymbol{r}'$, so that $G(\boldsymbol{r},0)=\delta(\boldsymbol{r}-\boldsymbol{r}')$. We recast the equation in the following form:

$$\partial_t G=\frac{\Theta}{2}[(d+1)r^2\nabla^2-2r^2\partial_r^2]G$$

$$=\frac{\Theta}{2}\{(d-1)[r^2\partial_r^2+(d+1)r\partial_r]+(d+1)\hat{L}^2\}G,$$

where $\hat{L}$ is the angular momentum operator in d dimensions defined as the angular part of $r^2\nabla^2$. Symbolically the solution is $e^{-(d+1)\hat{L}^2\Theta t/2}G_{\rm rad}$, showing that the PDF is getting isotropic after the characteristic time $1/[(d+1)\Theta]$. However, we are rather interested in the PDF of $|\boldsymbol{r}|$, $G(r,t)=r^{d-1}\int d\Omega\, G(|\boldsymbol{r}|=r,t)$. The angular momentum eigenfunctions, besides the zeroth one, give zero after angular integration. On the other hand, they evolve in time independently. Expanding the initial condition in the angular momentum eigenfunctions $G(\boldsymbol{r},0)=\Gamma(d/2)/[2\pi^{d/2}(r')^{d-1}]\delta(r-r')+\cdots$, where the dots stand for the terms vanishing after angular integration, we conclude that $G(r,t)$ is given by

$$G(r,t)=\frac{1}{2\sqrt{\pi Dt}}\frac{r^{d-1}}{r'^{d}}\exp\left[-\frac{\left(\ln\frac{r}{r'}+Ddt\right)^2}{4Dt}\right]. \tag{6}$$

Considering the statistics of the Lyapunov exponent defined by $\lambda\equiv(1/t)\ln r/r'$, we find

$$G(\lambda,t)=\frac{1}{2}\sqrt{\frac{t}{\pi D}}\exp-\frac{t(\lambda-Dd)^2}{4D}. \tag{7}$$

Thus we derived in a different way the result of [4] stating that the Lyapunov exponent is positive and possesses Gaussian statistics. Now we present the solution of Eq. (3). Rewrite it as

$$\partial_t P=\frac{B}{2}\nabla_w^2 P+\frac{\Theta}{2}\left((d+1)\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_i}w_j w_j-2\frac{\partial}{\partial w_i}\frac{\partial}{\partial w_j}w_j w_i\right)P.$$

Making the Fourier transform over $\boldsymbol{w}$, we have

$$\partial_t P=\frac{\Theta}{2}[(d+1)k^2\nabla_k^2-2k^2\partial_k^2]P-\frac{B}{2}k^2P. \tag{8}$$

Again we first consider the radial equation

$$\partial_t P=D[k^2\partial_k^2+(d+1)k\partial_k-w_*^2k^2]P. \tag{9}$$

D was defined before and $w_*^2\equiv B/\Theta(d-1)$ is the parameter defining the scale of gradients. To solve the above equation we introduce $f(k,t)\equiv(k/w_*)^{d/2}P(k/w_*,t)$. Then f satisfies

$$\partial_t f=D\left[k^2\partial_k^2+k\partial_k-\left(\frac{d^2}{4}+k^2\right)\right]f. \tag{10}$$

On the right-hand side one has a differential equation for modified Bessel functions. This suggests using the Kontorovich-Lebedev integral transform,

$$f(k,t)=\int_0^\infty dx\; g(x,t)K_{ix}(k), \tag{11}$$

$$g(x,t)=\frac{2}{\pi^2}x\sinh(\pi x)\int_0^\infty f(k,t)K_{ix}(k)\frac{dk}{k}, \tag{12}$$

where K_{ix} are the modified Bessel functions of the third kind. It gives the solution of the Cauchy problem for $P(k,t)$:

$$P=\frac{e^{-Dtd^2/4}}{k^{d/2}}\int_0^\infty dx\;\exp[-Dtx^2]K_{ix}(w_*k)g(x,0) \tag{13}$$

with $g(x,0)$ given by

$$g=\frac{2x\sinh(\pi x)}{\pi^2 w_*^{d/2}}\int_0^\infty dkP\left(\frac{k}{w_*},0\right)K_{ix}(k)k^{d/2-1}.$$

We first concentrate on the most typical case of small initial gradients of the passive scalar, for the investigation of which we take the initial condition $P(\boldsymbol{w},0)=\delta(\boldsymbol{w})$. That means that the typical value of initial gradients is much less than w_*. Then using the above solution and performing the inverse Fourier transform one finds

$$P(w,t)=\frac{e^{-Dtd^2/4}}{4\pi^{d/2+2}\Gamma(d/2)w_*^d}\int_0^\infty dx\; x\sinh(\pi x)\; e^{-Dtx^2}\times\left|\Gamma\left(\frac{d}{4}+\frac{\imath x}{2}\right)\right|^4F\left(\frac{d}{4}+\frac{\imath x}{2},\frac{d}{4}+\frac{\imath x}{2},\frac{d}{2},-\left[\frac{w}{w_*}\right]^2\right), \tag{14}$$

where $F(\alpha,\beta,\gamma,z)$ is the hypergeometric function. This is an exact solution whose asymptotics we now study. The general Green's function satisfying $P(w,0)=\Gamma(d/2)/[2\pi^{d/2}(w')^{d-1}]\delta(w-w')$ is obtained by adding $F(d/4+\imath x/2,d/4+\imath x/,2d/2,-[w'/w_*]^2)$ to the integrand above and in the limit $w_*\gg w'$ it reduces to Eq. (14) as was stated before. The asymptotic expansions are

$$P(w,t)\approx\frac{\Gamma^2(d/4)e^{-Dtd^2/4}}{8\pi^{(d+1)/2}w_*^d(Dt)^{3/2}}\left(\frac{w_*}{w}\right)^{d/2}\ln\left(\frac{w}{w_*d}\right),$$

$$w_*\ll w\ll w_*\exp(\sqrt{Dt}),\quad Dt\gg1, \tag{15}$$

$$P(w,t)\approx\frac{\Gamma^4(d/4)e^{-Dtd^2/4}}{16\pi^{(d+1)/2}w_*^d\Gamma(d/2)(Dt)^{3/2}}\times F\left(\frac{d}{4},\frac{d}{4},\frac{d}{2},-\left[\frac{w}{w_*}\right]^2\right),$$

$$w\ll w_*\exp(\sqrt{Dt}),\quad Dt\gg1, \tag{16}$$

$$P(w,t)\approx \frac{e^{-Dtd^2/4}}{4\pi^{(d+1)/2}w_*^d\sqrt{Dt}}\left(\frac{w_*}{w}\right)^{d/2}\frac{\Gamma^2\left(\frac{d}{4}+\frac{\ln(w/w_*)}{4Dt}\right)}{\Gamma\left(\frac{\ln(w/w_*)}{2Dt}\right)}$$

$$\times e^{-\ln^2(w/w_*)/4Dt},\quad Dt\gg 1,$$

$$e^{\sqrt{Dt}}\gg d,\quad \exp[2Dte^{\sqrt{Dt}}]\gg \frac{w}{w_*}\gg d, \tag{17}$$

$$P(w,t)\approx \frac{e^{-Dtd^2/4}}{w_*^d(2\pi)^{d/2}\sqrt{2Dt}}\left(\frac{w_*}{w}\right)^{d/2+\ln 2/2Dt}$$

$$\times\left(\frac{\ln(w/w_*)}{2Dt}\right)^{(d-1)/2}$$

$$\times e^{-\ln^2 2/4Dt}\exp\left[-\frac{\ln^2(w/w_*)}{4Dt}\right],$$

$$w\gg w_* e^{Dtd^2},\quad w\gg w_* e^d, \tag{18}$$

$$P(w,t)\approx(2\pi Bt)^{-d/2}\exp\left[-\frac{w^2}{2Bt}\right],$$

$$w\ll w_*\sqrt{d},\quad Dtd^3\ll 1. \tag{19}$$

The above asymptotics have a simple physical meaning. Large gradient statistics is velocity-determined. Indeed, the source injects passive scalar blobs of the characteristic size L, while advection creates filaments possessing large transverse gradients. Roughly, advection brings about random walk for gradients in the logarithmic scale, whereas the pumping does it in the usual scale. When we start from the uniform distribution of the scalar, then the advection is unimportant at the initial stage because it has nothing to mix. And indeed the main portion of the PDF for small times is purely pumping-determined, as asymptotics (19) reflects. Nevertheless, it should be stressed that the tails are always determined by the advection, as Eq. (18) (valid at any t) shows. They arise at $t=+0$ similarly to the usual diffusion equation, and are log-normal. This log-normality was first predicted by Kraichnan in [3]. How diffusion modifies them in the steady state is discussed in [1,5]. The above suggests the following picture. Large gradient ($w\gg w_*$) statistics is determined by the advection but it needs some initial nonvanishing gradient to start from. The latter is created by the source and naturally equals w_*, the ratio between pumping variance and the one of strain multiplied by $d-1$. The last factor arises because the role of advection grows with dimension. And, indeed, if one considers the solution of the equation without pumping with the initial value $w'=w_*$,

$$P(w,t)=\frac{1}{2\sqrt{\pi Dt}}\frac{\Gamma(d/2)}{2\pi^{d/2}w_*^d}\exp\left[-\frac{\left(\ln\frac{w}{w_*}+Ddt\right)^2}{4Dt}\right], \tag{20}$$

one observes that asymptotics (18), (17), and (15) are exactly its corresponding limits. However, there are subleading dependencies in our asymptotics [as, e.g., ln in Eq. (15)], which could not have been guessed on the basis of the above arguments. They arise as a result of the fluctuations of the effective initial distribution from which advection starts working. The most delicate result is Eq. (16) considered for $w\leqslant w_*$ [otherwise it goes to Eq. (15)]. This is exactly the region where the interplay between pumping and advection takes place and it cannot be explained by separating the influences. What should be emphasized is that it is analytic near zero and goes to a constant that monotonically decreases with time as $w\to 0$. When one includes diffusivity, the stationary PDF also goes to some constant as $w\to 0$, however it is nonanalytic at zero [1]. The last fact, most probably, should be attributed to the presence of highly diffused passive scalar sheets [3].

One of the main concepts of turbulence is the one of statistically restored symmetry [6]. The way the rotational symmetry is restored for the passive scalar problem can be explicitly shown within our approach. As we demonstrate, at large Pe (turbulent regime), isotropization mainly happens long before the diffusion becomes important and therefore is described by our Fokker-Planck equation. The isotropization is due to advection (pumping ''remembers'' direction). The analysis resembles that of two-point statistics. Consider Eq. (8). We see that again symbolically in Fourier space one has $e^{-(d+1)\Theta\hat{L}^2t/2}P_{\mathrm{rad}}$ showing that the time of the PDF's isotropization is of the order $t\sim 1/D$. As a particular example we consider the evolution of the ''slope'' that is a constant initial gradient in two dimensions with $P(\boldsymbol{w},0)=\delta(y)\delta(x-w')$. Introducing spherical coordinates in Fourier space,

$$P(\boldsymbol{k},0)=e^{ikw'\cos\theta}=J_0(kw')+2\sum_{n=1}\imath^n J_n(kw')\cos(n\theta).$$

We have for $g(x,0)$ defined in Eq. (13)

$$g=\frac{2}{\pi^2}x\sinh(\pi x)\int_0^\infty\left(\frac{k}{w_*}\right)^{d/2}J_0\left(\frac{kw'}{w_*}\right)K_{ix}(k)\frac{dk}{k}$$

$$+\frac{4}{\pi^2}\sum_{n=1}\imath^n e^{-n^2\Theta(d+1)t/2}\cos(n\theta)x\sinh(\pi x)$$

$$\times\int_0^\infty\left(\frac{k}{w_*}\right)^{d/2}J_n\left(\frac{kw'}{w_*}\right)K_{ix}(k)\frac{dk}{k}.$$

It is clear that our PDF evolution can serve as a good approximation to the one with diffusion only within some interval of gradients $[0,R(t)]$, with $R(t)$ being a decreasing function of time (eventually going to zero). If $R(1/D)\gg w_*$, then because the form of the PDF at large gradients is velocity-determined and thus should be isotropic, one may assert that isotropization happened before the diffusion became effective. However, the check if two PDF's are close near zero can be done by considering the equality of the moments produced by them. If we choose as a criterion that the first $2n$ moments do not differ appreciably, then the time at which this condition breaks should have the form $t=f(n)\ln(Pe)/D$, with $f(n)$ some decreasing function of n. This is due to the fact that the only possible characteristic time is $\ln(Pe)/D$, which is the period during which the blob of initial size L is stretched to the diffusion length. Thus by choosing large enough Pe (which corresponds to the turbulent regime) one can satisfy the above inequality. We conclude that isotropization is a property of the velocity field and in the turbulent regime happens long before the diffusion becomes effective. To make our words more tangible we consider the above criterion for $n=2$.

We find the critical time at which the ratio of the expectation values of $(\nabla\theta)^2$ with and without diffusion differs appreciably from 1. For this purpose, consider a two-point correlation function of the full passive scalar problem $H(\boldsymbol{r},t)=\langle\theta(\boldsymbol{r}_1,t)\theta(\boldsymbol{r}_1+\boldsymbol{r},t)\rangle$. It satisfies

$$\partial_t H=\frac{\Theta}{2}\boldsymbol{r}_j\partial_i[(d+1)\boldsymbol{r}_j\partial_i-\boldsymbol{r}_j\partial_i-\delta_{ij}\boldsymbol{r}_k\partial_k]H+2\kappa\nabla^2 H+\chi(r). \tag{21}$$

This equation is the same as the Fokker-Planck one with the change $B/2\to 2\kappa$ and the forcing term. Imposing the zero initial condition and using isotropy of χ one gets

$$H(r,t)=\int_0^t dt'\int_0^\infty dr'\mathcal{G}(r,r',t-t')\chi(r')$$

with $\mathcal{G}$ the Green's function found above but without the normalization factor and with the appropriate change in the definition of w_*^2. $\langle(\nabla\theta)^2\rangle=-\nabla^2 H(r=0)$ implies

$$\langle(\nabla\theta)^2\rangle=-\int_0^t dt'\int_0^\infty dr'\chi(r')\nabla_r^2\mathcal{G}(r,r',t-t')_{(r=0)}.$$

Directly using the equation satisfied by the Green's function one finds that $\nabla_r^2\mathcal{G}(r,r',t-t')_{(r=0)}=1/(2\kappa)\partial_t\mathcal{G}_{(r=0)}$. Using the symmetry of the Green's function in its spatial arguments and the fact that $\mathcal{G}(r=0,r',t)=2\pi^{d/2}r'^{d-1}/\Gamma(d/2)P(0,r',t)$, where P is the PDF of $\nabla\theta$ considered before but with w_* changed as described above, we obtain

$$\langle(\nabla\theta)^2\rangle=\frac{\chi(0)}{2\kappa}-\frac{2\pi^{d/2}}{2\kappa\Gamma(d/2)}\int_0^\infty dr'\chi(r')P(r',0,t)r'^{d-1}.$$

Looking at the way $P(r',0,t)$ depends on w_* we find

$$\int_0^\infty dr'\chi(r')P\left(r',0,t,w_*=\left[\frac{2\kappa}{D}\right]^{1/2}\right)r'^{d-1}=\int_0^\infty dr'\chi\left(\left[\frac{4\kappa}{B}\right]^{1/2}r'\right)P\left(r',0,t,w_*=\left[\frac{B}{2D}\right]^{1/2}\right)r'^{d-1}.$$

Note that now P is the PDF for $\nabla\theta$ without diffusion. Assuming that χ can be expanded in the Taylor series inside the integral, we obtain

$$\langle(\nabla\theta)^2\rangle_d=\langle(\nabla\theta)^2\rangle-\frac{1}{2\kappa}\times\sum_{n=2}\frac{\chi^{(2n)}(0)}{(2n)!}\left(\frac{4\kappa}{-[\chi^{(2)}(0)]}\right)^n\langle(\nabla\theta)^{2n}\rangle,$$

where we have used $B=-\chi^{(2)}(0)$ and the subscript d means an average in the full problem with diffusion. This is the expansion we need because initially the moments are small and the first term in the series dominates. Thus the critical time is found from the condition

$$\frac{\langle(\nabla\theta)^4\rangle}{\epsilon\langle(\nabla\theta)^2\rangle}\sim 1,\qquad \epsilon\equiv\frac{\chi(0)}{2\kappa}. \tag{22}$$

We introduced the stationary dissipation ϵ and estimated $\chi^{(2n)}(0)\sim\chi(0)/L^{2n}$, which is good for the first derivatives of χ. Integrating both sides of the radial equation for PDF one can establish

$$\frac{1}{D}\partial_t\langle(\nabla\theta)^{2n}\rangle=(2n)(2n+d)\langle(\nabla\theta)^{2n}\rangle+(w_*)^2(2n+d-2)(2n)\langle(\nabla\theta)^{2n-2}\rangle.$$

Using the normalization of P this in principle lets one find recursively all the moments. Solving it for $n=1$ and then for $n=2$ one finds that Eq. (22) implies

$$t_{cr(2)}\sim\frac{1}{(d+6)D}\ln Pe, \tag{23}$$

where the index 2 refers to the estimation by the second moment. The time is proportional to $\ln(Pe)/D$.

To conclude, large gradient statistics is velocity-determined and, thus, log-normal [3], the role of pumping being the creation of the effective initial gradient for the velocity to mix. In the region of the small gradients an interplay between pumping and velocity takes place with the resulting analytic function going to some constant for zero gradient. In the turbulent regime of large Peclet numbers the isotropization of the initial asymmetry in the passive scalar distribution happens long before the diffusion becomes effective.

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